Rutgers University: Algebra Written Qualifying Exam

January 2016: Problem 5 Solution

Exercise. Let G be a group of order 108. Prove that G is not simple.

Solution.

$|G| = 108 = 2^2 3^3$, so by the third Sylow Theorem, $n_3 \equiv 1 \mod 3$ and $n_3 \mid 4$

$\implies n_3 = 1$ or 4

If $n_3 = 1$ then there's one Sylow 3-subgroup, which is normal in G by the 2^{nd} Sylow Theorem.

$\implies G$ is not simple.

Suppose there are $n_3 = 4$ Sylow 3-subgroups, and let H, K denote distinct Sylow 3-subgroups.

Want to show $H \cap K \lhd G$

1. Find the **order** of $H \cap K$

$$|HK| = \frac{|H||K|}{|H \cap K|} \implies |H \cap K| = \frac{|H||K|}{|HK|} \geq \frac{|H||K|}{|G|} \text{ since } |HK| \leq |G|$$

$$= \frac{27 \cdot 27}{27 \cdot 4} = \frac{27}{4} = 6.75$$

Moreover, $H \cap K$ is a subgroup of H and K

$\implies |H \cap K| \mid |H| = 3^3$

$\implies |H \cap K| = 3$ or 9

Since $|H \cap K| \geq 6.75$, we know $|H \cap K| = 9$.

2. Find the **index** of $H \cap K$ to show $H \cap K \lhd H$ and $H \cap K \lhd K$

$$|H| = [H : H \cap K]|H \cap K|$$

$$\implies 27 = [H : H \cap K] \cdot 9$$

$$\implies [H : H \cap K] = 3$$

Similarly, $[K : H \cap K] = 3.$

If a subgroup hast the smallest prime index, then it is normal.

Since 3 is the smallest prime divisor of $27 = |H| = |K|$, $|H \cap K| \lhd H$ and $H \cap K \lhd K$.

3. Use $N_G(H \cap K) = \{g : g(H \cap K)g^{-1} = H \cap K\}$ to show $H \cap K \lhd G$

By part (2), $H, K \subseteq N_G(H \cap K)$, so

$h(H \cap K)h^{-1} = H \cap K, \ \forall h \in H$ and $k(H \cap K)k^{-1} = H \cap K, \ \forall k \in K$

$$\implies (hk)(H \cap K)(hk)^{-1} = hk(H \cap K)k^{-1}h^{-1}$$
$$= h(H \cap K)h^{-1}$$
$$= H \cap K, \ \forall hk \in HK$$

Thus, $HK \subseteq N_G(H \cap K)$

$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{27 \cdot 27}{9} = 81$$

$N_G(H \cap K)$ is a subgroup of G so $|N_G(H \cap K)| \mid |G|$

$\implies |N_G(H \cap K)| \mid 27 \cdot 4$

$\implies |N_G(H \cap K)| = 108 = |G|$ since $|N_G(H \cap K)| \geq 81$

Thus, $N_G(H \cap K) = G$ and so $H \cap K \lhd G$.